Torsional Oscillations of a Rotating Column of ^{3}He-B

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We have analysed the axisymmetric and non-axisymmetric modes of a continuum of vortices in a rotating superfluid. We have investigated how changing the temperature affects the growth rate of the disturbances. We find that, in the long axial wavelength limit the condition $q = \alpha/(1-\alpha') = 1$, where α and α' are temperature-dependent mutual friction parameters, is the crossover between damped and propagating Kelvin waves. Thus at temperatures for which $q > 1$, perturbations on the vortices are unlikely to cause vortex reconnections and turbulence. These results are in agreement with the recent discovery of Finne et al[1] of an intrinsic condition for the onset of quantum turbulence in ^{3}He-B.

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1. MODEL

When superfluid helium is rotated, an array of quantised vortex lines appear which are aligned parallel to the axis of rotation. This array exhibits oscillation modes which were first predicted and observed by Hall[2]. The modes are similar to classical inertial waves in the long wavelength limits and are related to the waves of isolated vortex lines in the short wavelength limit. These vortex waves are a very important phenomena in the understanding of quantised vortex lines and for a review of work in this area see Donnelly[3].

The aim of this paper is to extend previous work[4] to consider the torsional oscillations of a rotating column of superfluid at a range of temperatures. We consider ^{3}He-B rotating with angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_z$ in a cylinder of radius $r = a$. The basic state will consist of a uniform vortex

lattice with a large density of vortices aligned along the direction of rotation. In this context the behaviour of the superfluid is described by the Hall-Vinen equations[5]. Thus, rather than individual vortex lines, we consider a continuum of vortex lines. The advantage of this model is that it allows us to explore effects which the theory of a single vortex filament[6] cannot describe, notably the presence of boundaries and the degrees of freedom represented by the coherent oscillatory motion of many vortices.

The equations of motion of the superfluid in a coordinate system rotating with angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$ may be written as

$$\begin{aligned} \frac{\partial \boldsymbol{v}^s}{\partial t} + (\boldsymbol{v}^s \cdot \boldsymbol{\nabla})\boldsymbol{v}^s = \boldsymbol{\nabla}\Psi + 2\boldsymbol{v}^s \times \boldsymbol{\Omega} + \alpha\widehat{\boldsymbol{\lambda}} \times [\boldsymbol{\lambda} \times (\boldsymbol{v}^s - \boldsymbol{v}^n)] \\ +\alpha'\boldsymbol{\lambda} \times (\boldsymbol{v}^s - \boldsymbol{v}^n) - \alpha\nu^s\widehat{\boldsymbol{\lambda}} \times (\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})\widehat{\boldsymbol{\lambda}} + \nu^s(1-\alpha')(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})\widehat{\boldsymbol{\lambda}} \end{aligned} \tag{1}$$

where $\boldsymbol{v}^s$ and $\boldsymbol{v}^n$ are the superfluid and normal fluid velocities in the rotating frame, $\boldsymbol{\lambda} = \boldsymbol{\nabla} \times \boldsymbol{v}^s + 2\boldsymbol{\Omega}$, $\widehat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}/|\boldsymbol{\lambda}|$ is the unit vector in the direction of $\boldsymbol{\lambda}$ and Ψ is a collection of scalar terms. Given the high viscosity of ^{3}He-B, we assume that the normal fluid is in solid body rotation around the z-axis, thus in the rotating frame $\boldsymbol{v}^n = 0$. Equation (1) must be solved under the condition that $\boldsymbol{\nabla}\cdot\boldsymbol{v}^s = 0$. The quantity $\nu^s = (\Gamma/4\pi)\log(b_0/a_0)$ is the vortex tension parameter, Γ is the quantum of circulation, a_0 is the vortex core radius and $b_0 = (|\boldsymbol{\lambda}|/\Gamma)^{-1/2}$ is the average distance between vortices. The unperturbed vortex lattice corresponds to the basic state $\boldsymbol{v}_0^s = \boldsymbol{v}_0^n = 0$, $\nabla\Psi_0 = 0$ for which $\boldsymbol{\lambda} = 2\Omega\mathbf{e}_z$, working in cylindrical coordinates (r, ϕ, z).

We perturb the basic state by letting $\boldsymbol{v}^s = \boldsymbol{u} = (u_r, u_\phi, u_z)$, $\Psi = \Psi_0 + \psi$, where $|\boldsymbol{u}| \ll 1$ and $|\psi| \ll 1$ and linearise the resulting equations of motion. We assume normal modes for the perturbations of the form $(\boldsymbol{u}, \psi) = (\widehat{\boldsymbol{u}}(r), \widehat{\psi}(r))\exp(\imath\sigma t + \imath m\phi + \imath kz)$ where m and k are respectively the azimuthal and axial wavenumbers and σ is the growth rate. The aim of our calculation is to determine the real and imaginary parts of σ, namely $\mathcal{R}e(\sigma)$ and $\mathcal{I}m(\sigma)$.

The solution for $\widehat{u}_z$ which is regular as $r \to 0$ is the Bessel function of the first kind of order m, $J_m(\beta r)$, where

$$\beta^2 = \frac{-k^2[(\imath\sigma + \alpha\eta)^2 + (1-\alpha')^2\eta^2]}{[(\imath\sigma + \alpha\eta)(\imath\sigma + k^2\nu^s\alpha) + (1-\alpha')^2\nu^s k^2\eta]} \tag{2}$$

and $\eta = 2\Omega + \nu^s k^2$. To determine β we enforce the boundary condition $u_r = 0$ on the wall of the container $r = a$, which yields the secular equation

$$\frac{(ka)^2}{(\beta a)}\frac{J_m'(\beta a)}{J_m(\beta a)}[(\imath\sigma + \alpha\eta)^2 + (1-\alpha')^2\eta^2] + 2m\Omega\sigma(1-\alpha') = 0, \tag{3}$$

where the prime denotes the derivative of the Bessel function with respect to its argument.

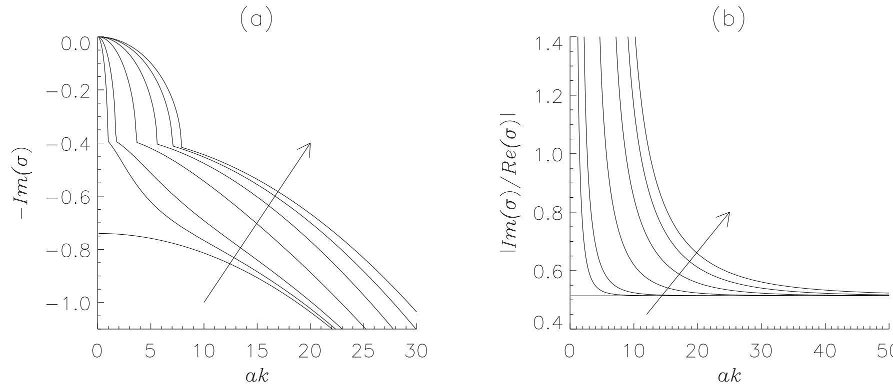

Fig. 1. Plots of (a) the growth rate, $-\mathcal{I}m(\sigma)$ (b) $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|$ of the axisymmetric mode ($m = 0$) against ak for various values of ξ_j at $T/T_c = 0.5$. The arrows show the direction of increasing ξ_j, where $j = 0, 1, 2, 5, 10, 20, \infty$.

2. RESULTS

Equations (2,3) simplify in the axisymmetric case, $m = 0$. From (3) we find that β must satisfy $J_0'(\beta a) = J_1(\beta a) = 0$. Taking ξ_j to be the j^{th} zero of $J_1(\xi) = J_0'(\xi)$ (that is $\xi_0 = 0$, $\xi_1 = 3.83171$, $\xi_2 = 7.01559$ *etc*), Equation (3) becomes

$$\frac{\xi_j^2}{(ka)^2} = -\frac{[(\imath\sigma + \alpha\eta)^2 + (1-\alpha')^2\eta^2]}{[(\imath\sigma + \alpha\eta)(\imath\sigma + \alpha\nu^s k^2) + (1-\alpha')^2\nu^s\eta k^2]}. \tag{4}$$

This quadratic equation may be solved analytically[7,4] and has an infinite number of solutions, according to the value of ξ_j considered. All the solutions are such that $\mathcal{I}m(\sigma)$ is non-negative, so the system is always stable to infinitesimal disturbances. The least stable mode is the one for which $\mathcal{I}m(\sigma)$ is minimum and is given by

$$\sigma = \imath\alpha(\Omega + \nu^s k^2) \pm \sqrt{\{(1-\alpha')^2\nu^s k^2(2\Omega + \nu^s k^2) - \alpha^2\Omega^2\}} \tag{5}$$

which corresponds to $\xi_j \to \infty$.

Plots of the growth rate, $-\mathcal{I}m(\sigma)$ and the ratio of the decay rate of the wave to its angular frequency, $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|$ can be found in Fig. 1, where the mutual friction parameters correspond to $T = 0.5T_c$ for which $\alpha = 0.38$ and $\alpha' = 0.28$, where T_c is the critical temperature and we have taken $\nu^s/a^2 = 0.002$ and $\Omega = 1$. The kinks in the curves of Fig. 1(a) for $j > 0$ correspond to σ switching from being purely imaginary (for which the decay of the mode is monotonic with time) to complex (for which the decay

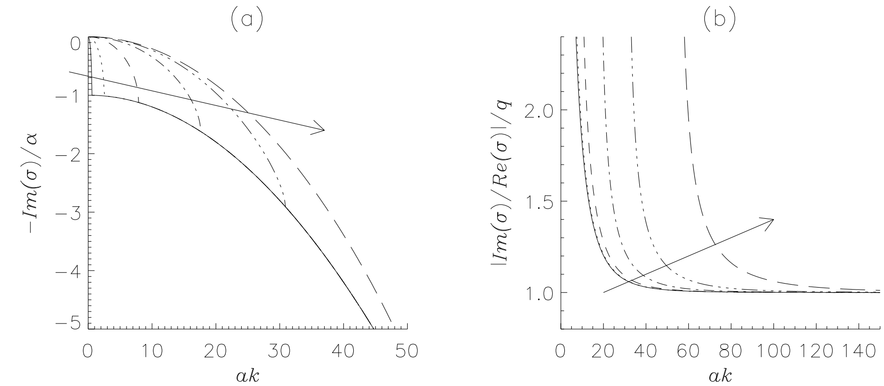

Fig. 2. Plots of (a) $-\mathcal{I}m(\sigma)/\alpha$ (b) $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|/q$ of the least stable axisymmetric mode against ak for various temperatures. The arrows show the direction of increasing T, where $T/T_c = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$.

of the mode is oscillatory). For the least stable mode, the crossover occurs at $k = k_*$ where

$$k_*^2 = \frac{\Omega}{\nu^s}(-1 + \sqrt{1 + q^2}) \tag{6}$$

where $q = \alpha/(1 - \alpha')$. The parameter q increases rapidly with temperature[8] and therefore k_* also increases rapidly with temperature. This effect is illustrated in Fig.2(a) in which, for clarity of scale, we plot $-\mathcal{I}m(\sigma)/\alpha$ against (ak) of the least stable axisymmetric mode at various temperatures. The arrow shows the direction of increasing temperature. For $k > k_*$, the expression in the square root of (5) is positive so the normalised growth rate is independent of temperature and is given by $\mathcal{I}m(\sigma)/\alpha = \Omega + \nu^s k^2$. The mutual friction parameter α increases rapidly with temperature so the axisymmetric modes will decay much faster at higher temperatures than at lower temperatures. In Fig.1(b) we plot the ratio of the decay rate of the wave to its angular frequency, $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|$ against (ak) We see that for all the modes $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)| \to q \approx 0.51$ at large (ak). In Fig.2(b) we plot $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|/q$ against (ak) of the least stable axisymmetric mode at various temperatures. Again we find $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)| \to q$ at large (ak) for all temperatures and that this limiting value is obtained at larger values of (ak) as the temperature is increased.

In order to consider the non-axisymmetric modes we must solve the coupled equations (2,3) and further details of the solution method can be found in Henderson & Barenghi [4]. In Fig. 3 we plot $-\mathcal{I}m(\sigma)$ and $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|$ against ak of the least stable computed modes for $m = 0, 2, 4, 7, 10$ using the

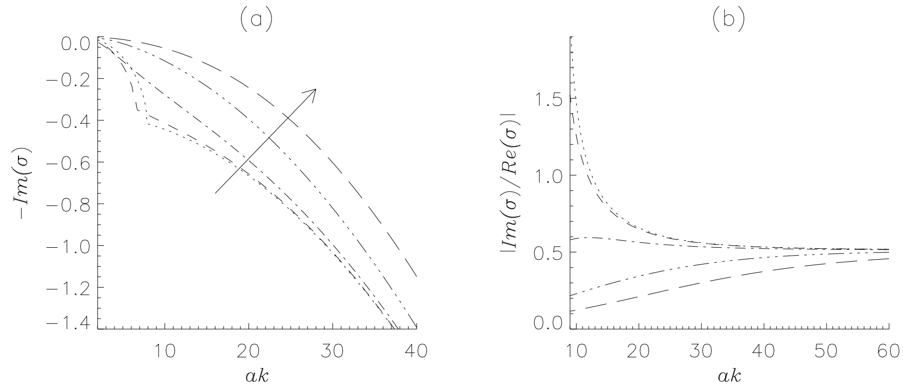

Fig. 3. Plots of (a) the growth rate, $-\mathcal{I}m(\sigma)$ (b) $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)|$ against ak for the $m = 0, 2, 4, 7, 10$ mode. The arrow shows the direction of increasing m.

same parameters as for Fig. 1. The arrow shows the direction of increasing m. We do not plot the $m = 1$ for clarity; it suffices to say that the least stable $m = 1$ mode is bound by the least stable $m = 0$ mode.

It can be seen that the modes become less stable as m is increased which is similar to what we found at lower temperatures[4]. We can also see that at large (ak), all the modes are such that $|\mathcal{I}m(\sigma)/\mathcal{R}e(\sigma)| \to q \approx 0.51$ and that this limit is approached from below for larger values of m. What this means is that oscillatory decay is more pronounced for modes with larger azimuthal wavenumbers, particularly at small to moderate values of the axial wavenumber. As for the azimuthal mode, we find that the decay rates of the azimuthal modes, $\mathcal{I}m(\sigma)$ are proportional to α, so at higher temperatures, for which α is small, the mode will decay more rapidly in time.

3. APPLICATION

Our work has applications to the recent discovery by Finne *et al*[1] of an intrinsic condition for the onset of quantum turbulence in ^{3}He-B, namely $q > 1.3$. Finne *et al* interpreted their result in terms of Kelvin waves - helical perturbations of the position of a vortex core away from the unperturbed straight shape. In the case of an isolated vortex line, it was predicted [6] that $q = 1$ is the crossover between Kelvin waves which propagate $(q < 1)$ and Kelvin waves which are damped $(q > 1)$. If $q < 1$ the propagating Kelvin waves grow in amplitude exponentially with time, driven by the local

difference between the normal fluid velocity and the velocity of the vortex line.

If this condition is valid for a large number of vortices (which have more degrees of freedom that a single vortex line), then one expects that, when the waves' amplitude becomes of the order of the average intervortex spacing, the vortices reconnect which each other, and form a turbulent tangle.

Our analysis shows that, for both axisymmetric and non-axisymmetric perturbations of a vortex lattice, the ratio of imaginary and real part of the complex growth rate is approximately equal to the parameter q identified by Finne *et al* in the large axial wavenumber limit, for all temperatures. The minimum permitted value of k will be governed by the height, h of the apparatus by the relation $k_{\rm min} = 2\pi/h$, so provided that the aspect ratio is small enough ($h/a \ll 1$) we find agreement with the argument of Finne *et al*. Therefore if $q > 1$, perturbations shrink in amplitude before they can rotate a full cycle (overdamping); this reduces the volume of space which is swept by the vortex lines in their motion, which reduces the probability of making reconnections with neighbouring vortices, and without reconnections there is no turbulence. Vice versa, if $q < 1$ the perturbations propagate, which favours vortex reconnections, hence turbulence.

Finally, the exact dispersion relation which we have found (2,3) can be used to study with precision the torsional oscillations of on a rotating vortex lattice for any height and radius of experimental apparatus, which is a topic of current experimental interest[9].

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